The Nature of Time and the Structure of Space

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1 Project Abstract

My application is for two mutually reinforcing projects. The first is to show that the structure of space essentially determines the dynamics of space, which in turn determines the physcial properties of time. This will be done by completing my program for the relational derivation of classical dynamics from the fewest possible axioms. In particular, only scale-invariant (angle-determining) structure of space is presupposed. Much of the structure of spacetime usually taken as fundamental is thereby shown to be emergent.

This is likely to be important in quantum gravity, in which emergent structure of the classical theory should play no fundamental role. My second project is to write a monograph presenting a unifying vision of the relational foundations of physics. I am confident that I do now have a clear overview of relationalism in classical dynamics. The part played by scale invariance – the relativity of size and its relation to time – was the last piece of the picture to fall into place. A monograph that presents this picture will have value in itself and be a resource for researchers wishing to apply the insights of relational dynamics in quantum gravity.

2 Summary for Laypeople

My research project addresses the most fundamental questions in dynamics: What is space? What is time? What is motion? They were hotly debated by Newton and Leibniz three centuries ago and still have central importance because they have to be reconsidered with each new advance in our understanding of nature. They are critical for the greatest outstanding problem in physics: the unification of Einstein's general theory of relativity with quantum mechanics in order to create a quantum theory of the universe. Such a theory is needed to explain why the universe exists in the form it does, why it seems to have been created in a big bang, and why "never resting time" seems always to flow forward from past to future through an elusive present. For many years, I have studied the foundations of dynamics and have shown that Einstein's theory of relativity answers the questions as to the nature of space, time, and motion in a manner that has not hitherto been fully appreciated. My project has two aims: to bring this study to its conclusion and to summarize all this work in a book written to help the creation of the quantum theory of the universe.

3 Introduction and Overview

The main aim of my research project is to show that the structure of space essentially determines the dynamics of space, which in turn determines the physcial properties of time, i.e., the nature of time. This will be done by completing my program for the Machian (relational) derivation of the classical dynamics of the universe from the fewest possible axioms. These are: 1) minimal assumptions about the structure of space and the framework it provides for the description of matter; 2) specification of what initial data should determine dynamical evolution; 3) the basic mechanism (best matching) through which this is realized.

I see this project, which should bring to completion my current FQX-funded research, as important preparation for the creation of a corresponding quantum theory, the unfulfilled dream of physicists for at least half a century.

The main difficulty here has been reconciliation of the foundational axioms of general relativity and quantum theory. They are incompatible, above all in treating time in radically different ways. For several decades, I have been developing a relational interpretation of general relativity inspired by Mach's critique of Newton's concepts of absolute space and time. Although Mach's principle was the main stimulus to Einstein's creation of general relativity, his indirect approach to the task has been the source of much confusion about the essential dynamical structure of his great theory. Einstein himself even disowned Mach's principle at the end of his life, declaring that it had been made obsolete by the creation of field theory.

In fact, the work of my collaborators and myself [1, 2, 3, 4, 5] has shown that Einstein was quite wrong to draw this conclusion, which was based on an incomplete understanding of the true nature of Mach's profound critique [6]. If one considers properly how the fundamental principles of dynamics need to be formulated in order to do justice to the critique, one finds that general relativity is in fact a perfectly constructed Machian theory. However, this is hidden in a subtle manner within the mathematical formalism that Einstein took over 'ready made' in the form of the tensor calculus developed by Christoffel, Levi-Civita, and Ricci. When that is 'unpacked' and interpreted in a truly relational manner, general relativity appears in a very different light. Moreover, its structure no longer looks so incompatible with the most basic features of quantum theory: the common foundation of both theories is one and the same relational configuration space.

Before describing the specific issues that I wish to research with my collaborators in a new grant, I need to provide the necessary background, and will therefore outline the major change that the relational interpretation brings about. This will help to identify the few outstanding issues that need to be resolved before this program can be regarded as completed. This background will also serve to explain why I wish to devote half of the funds for which I am applying to the writing of a book that will give a complete account of the relational approach to classical and quantum mechanics. It will in effect be the second volume of my study of the great foundational question in dynamics: is motion absolute or relative? The first volume appeared in 1989 as *The Discovery of Dynamics* [7] and covered the period up to Newton's discovery of the laws of motion. My proposed second volume – an outline of its contents is given after my proposed research topics have been listed – will be less historical and much more concerned with cutting-edge research. It will be a complete reformulation of dynamics on a relational basis, as was already called for by Leibniz and later Mach. The demonstration that, without realizing it, Einstein in fact achieved precisely what they wanted is surely of great importance for the next major advance: quantum gravity.

3.1 General Relativity in the Spacetime Formulation

The most fundamental premise on which general relativity is based is the Minkowskian nature of spacetime in the infinitely small. Time and space are treated as almost identical in nature: distance is extension in a spacelike direction, duration is extension in a timelike direction. Only the opposite sign of the metric distinguishes the time dimension from space. Moreover, distance and duration are ontological: they exist in the world. Minkowski spacetime also has inbuilt inertial frames of reference, since test particles are forced to move in it along straight lines. To summarize, local inertial frames of reference, local proper distance and local proper time all belong to the fundamental kinematic structure of spacetime even though it is in general curved in the large. All possible spacetimes have these kinematic elements. The physically realized spacetimes, i.e., those that satisfy Einstein's field equations, are selected among the set of all possible spacetimes by an extremal property: the variation of

$$\int \mathrm{d}^4x\sqrt{g}R, \tag{1}$$

where R is the four-dimensional scalar curvature and g is the determinant of the 4-metric $g_{\mu\nu}$, must vanish for all variations of the metric in any region of spacetime with fixed values of the metric on the boundaries of the region.

This way of deriving Einsteinian spacetimes makes it appear that local inertial frames, local proper time, and local proper distance *really do exist* in the world, certainly at the classical level. There is therefore a natural presumption that they should be 'taken over' into the quantum domain. Already more than 50 years ago this assumption had to be revised when Dirac and Arnowitt, Deser, and Misner (ADM) cast general relativity into Hamiltonian form as a first step to its canonical quantization. As Wheeler emphasized, the first 'painful' lesson that had to be learned is that in Hamiltonian general relativity it is not the four-metric $g_{\mu\nu}$ of spacetime that evolves but the three-metric g_{ij} of *space*. Dirac was so struck by this discovery, which he welcomed for the *simplification* it introduced in relativistic dynamics, that he commented "This result has led me to doubt how fundamental the four-dimensional requirement in physics is." A few years later, DeWitt made the equally remarkable discovery that the canonical quantum theory of a closed unverse is *static*. Time disappears entirely.

3.2 My Long-Term Research Program

These major developments in quantum gravity explain how my own research program began. In 1963 I chanced to read Dirac's sentence in a newspaper article. At the same time I became acquainted with Mach's writings and was also studying the papers in which Einstein created general relativity in order to implement Mach's principle. Surprised by Einstein's failure to attack the Machian proposal directly, I resolved to go back to 'Machian first principles' in order to derive a fully relational theory *ab initio*. In such a theory, duration should be derived from change,[1] position should be defined relative to the universe at large, and only relative sizes within the universe should have objective meaning (if all scales were doubled overnight, no change could be observed). Of these three principles, the *axioms* of a truly relational theory, Einstein had consciously worked to implement just the second, the relativity of position and then, as noted, only indirectly.

I formulated a basic framework in which these axioms could be implemented in 1974 [9]. This led to an extended collaboration with Bruno Bertotti and our paper in 1982 [1], in which we had important help from Karel Kuchař. This paper, which has become reasonably well known in the relativity and quantum gravity communities, developed a universal framework in which one can automatically construct theories meeting the three essential relational requirements listed above. It is important that the relativity of duration is achieved in a manner that is quite different from that used to implement relativity of position and size, which is done by a process called *best matching* (for the basic idea, see [10]). In contrast, time is eliminated entirely from the basic kinematical structure of the theory, which is formulated as a *geodesic theory on configuration space.*

The paper [1] showed that in the case of a closed universe general relativity fully meets the requirements of relativity of duration and position. It also showed that the discoveries of Dirac, ADM, and DeWitt were simple direct consequences of these relational aspects of general relativity, which had long remained hidden in its original spacetime formulation. In particular, the disappearance of time in canonical quantum gravity discovered by DeWitt is a direct consequence of the relativity of duration. I also became convinced that the quantum implications of the relational structure of general relativity were more far reaching than had hitherto been appreciated and could perhaps explain the arrow of time and lead to a complete explanation of our sense of a passage of time in an ontologically timeless universe. I eventually wrote about this in my book *The End of Time* [11].

However, it now seems to me that the quantum implications are still more significant. This has to do with the relativity of size, which had not been further explored in the paper with Bertotti [1]. In 1999 I showed how this could be done for particles in Euclidean space [12], and the extension to dynamical geometry was then accomplished, initially by Niall Ó Murchadha and myself [13] and then in more detail by us with Edward Anderson, Brendan Foster, and Bryan Kelleher [3, 4]. The background to this work will be discussed in the next subsection, but I will conclude this subsection by

[1] "It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive by means of the changes of things" [8], p. 273. Einstein explicitly introduced relativity of simultaneity *but not of duration.*

noting a surprising bye product of the work on scale invariance. It transpired in the initial stages of this work that, if implemented locally, the Machian requirement that duration be derived from change is a immensely powerful principle and leads, in conjunction with relativity of position and a natural simplicity assumption, very directly to, first, general relativity, *then* special relativity, *and then* gauge theory [2]. The historical sequence of the discovery of these theories is reversed and seen in a very different light.

3.3 Weyl's Critique of Riemannian Geometry

Equally surprising, if not more so, were the insights that emerged from the relativity of size, the third axiom of relational dynamics. In the context of dynamical geometry, this means starting from the assumption that only angles and length ratios, and not lengths as such, play any dynamical role: scale has no more physical significance than coordinates in standard differential geometry. To implement this basic assumption, one seeks a theory that is *conformally covariant*.[2] The first attempt to create such a theory was made over 90 years ago by Weyl, who tried to dispense with one of the assumptions made by Riemann in 1854 when he created Riemannian geometry, namely that widely separated measuring rods can have the same physical length. Weyl argued that such global comparison of lengths is incompatible with the consistent development of geometry based on a purely local infinitesimal basis. To rectify the perceived defect, he assumed that the four-dimensional metric $g_{\mu\nu}(x,t)$ is determined only up to a scale factor $\phi(x,t)$ and thus determines only *angles* but not lengths in spacetime. He introduced a further field, a four-vector $\psi_\mu(x,t)$, to determine infinitesimal lengths. Just as the four-metric $g_{\mu\nu}$ and ψ_μ transform under coordinate transformations, both transform as well under scale transformations:

$$g_{\mu\nu} \to \phi g_{\mu\nu}, \ \psi_\mu \to \psi_\mu + \frac{\partial \phi}{\partial x_\mu}. \tag{2}$$

Specification of some $g_{\mu\nu}(x,t)$ then determines lengths only in a particular 'gauge',[3] which is changed by a transformation (2). Because the field ψ_μ transforms in the same way as the Maxwell vector potential A_μ, Weyl initially believed that he had succeeded in unifying gravitation and electromagnetism, but this attempt, which involved finding field equations invariant simultaneously under coordinate and gauge transformations, failed. The theory was nevertheless greatly admired for its critcal analysis of the problem of length determination. Moreover, after a small but significant change of the gauge group (leading from a real to a complex scale factor), Weyl's

[2]In differential geometry, conformal transformations change the scale of the metric, and in a conformally covariant theory scale cannot play a dynamical role.

[3]The notion and terminology of a gauge transformation (*Umeichung*) were introduced by Weyl in his 1918 paper.

idea became the basis of modern gauge theory. Of more relevance for the present research proposal, the original idea has never ceased to fascinate, and many researchers, Dirac included, have attempted to implement the idea in some form or other. Weyl's critique of Riemannian geometry clearly has great significance for the relational program and matches Leibniz and Mach's critique of Newton's absolute space.

3.4 Alternative Implementation of Weyl's Idea

It is in this light that the line of research begun by Ó Murchadha and myself in 1999 [13] is to be seen. It took into account the lessons learned in the early work on canonical quantum gravity and the complementary research of Bertotti and myself [1]. As already noted, this had above all demonstrated two things: first, the dynamical object in geometrodynamics is not the four-dimensional metric $g_{\mu\nu}$ but the three-dimensional metric g_{ij}; second, not all structure that appears in the spacetime form of Einstein's theory is necessarily ontological, it may be emergent. Our basic assumption was therefore this: *geometrodynamics is to be formulated in terms of the three-dimensional metric of space g_{ij}, but only its angle-determining part is to be regarded as physical.* Its scale factor, which determines local proper distance, is to emerge from the relational form of its dynamics just as local inertial frames of reference and local proper time arose in the earlier work from the relativity of position and duration.

Thus, our inspiration was the same as Weyl's but we attempted to implement it with significantly less fundamental structure; we dispensed entirely with Weyl's additional vector field ψ_μ and took only the conformal (angle-determining) spatial part of the spatial metric g_{ij}. We also significantly modified the invariance requirements imposed in Einstein's original spacetime formulation of general relativity and in Weyl's attempted generalization. Einstein required four-dimensional general covariance (involving four arbitrary functions of spacetime in the transformation laws, or spacetime diffeomorphism invariance), Weyl required not only that but also four-dimensional conformal invariance (a fifth arbitrary function). In contrast, we retained *three-dimensional* diffeomorphism invariance, which implements relativity of position, and replaced Einstein's freedom in the definition of simultaneity (foliation invariance) by three-dimensional conformal invariance, which ensures (local) relativity of size. Thus, with Einstein we required invariance with respect to four arbitrary functions but swapped one of them (foliation invariance) for three-dimensional conformal invariance.

Ó Murchadha succeeded in implementing these principles in a remarkable manner that generalizes standard gauge theory by relaxing some of the boundary conditions imposed on the variation.[4] The details are given in

[4]If one requires stationarity of an action subject to *weaker* restrictions on the variation (as in the free-end-point variation employed in [13, 3, 4]), then *stronger* restrictions on

[13, 3, 4]. Much of the research that I have been involved in during the second year of my current large grant has been devoted to clarifying and taking further this work. Several papers by my collaborators (Ó Murchadha, Sean Gryb, Henrique Gomes, Tim Koslowski) and myself are currently nearing completion. I shall not attempt to describe everything in detail, but merely highlight the most interesting and promising results.

4 Results Achieved with my Current Grant

First and foremost, the ontology of general relativity (for a closed universe at least) is radically changed. It becomes a theory of the dynamical evolution of *the shape of space*. The physical degrees of freedom at each space point are just the two angle-determining (conformal) parts of the three-metric g_{ij}, which describe the local shape of space. The corresponding configuration space is *conformal superspace* (CS), the space of all conformal three-geometries on a closed manifold M. We require that in a relational theory of conformal geometry *a point and a tangent vector* in CS should determine a unique evolution in CS. This is achieved in a theory that implements relativity of position and size by appropriate best matching. One of the issues that we are currently researching (see below) is how many theories exist which meet the above 'point-and-tangent-vector' requirement. However, it has already been established that general relativity is such a theory. In principle, this was already shown in [4], but the recent research has significantly strengthened the derivation, on which Ó Murchadha and I have just posted a paper [5].

We are also gaining a deeper understanding of the manner in which relativity of simultaneity (foliation invariance) can be 'swapped' for relativity of size (conformal invariance). We are beginning to see how the interpretation of general relativity as the *dynamics of shape* casts light on its deep dynamical structure. Ever since the work of York and Ó Murchadha 40 years ago, the importance of conformal techniques in the solution of the initial-value problem of general relativity has been well known. However, that work, done at a time when the primacy of foliation invariance (relativity of simultaneity) was unchallenged, was not felt to be relevant to dynamical evolution. Our work is beginning to cast doubt on that assumption, long felt to be the rock on which general relativity is founded.

In this connection, it is worth noting that our approach based on relativity of size is not the only one that questions the primacy of foliation invariance. For several years, researchers in quantum gravity have expected that the existence of the Planck length as a fundamental unit on a par with the speed of light will lead to a breakdown of strict Lorentz invariance at the Planck length (doubly special relativity). More recently, Hořava has ar-

the possible solutions are obtained. This is the key to our new results.

gued that foliation invariance is very strongly broken in the ultraviolet limit of perturbative quantum gravity. His approach, which has attracted huge interest, postulates in a rather *ad hoc* manner a distinguished foliation in spacetime. As will be noted in the main text, one of the most interesting results of our conformal approach is that it leads with iron necessity to a distinguished foliation. It follows from a fundamental first principle.

The possibility that relativity of simultaneity is not a true first principle for general relativity and needs to be replaced at the deepest level by relativity of size is intimately related to the manner in which Mach's dictum on the derivation of time from change (footnote 1) is implemented. As this is an integral part of the demonstration that the structure of space essentially determines its dynamics, which in turn determines the properties of time, the following subsections explain how this happens.

4.1 The Derivation of Time with Euclidean Geometry

Time emerges from timeless dynamics in a more or less universal manner, but the details depend crucially on the assumptions made about the structure of space. This will be outlined in this and the next two subsections for the progressively more interesting and far reaching cases of 1) Euclidean, 2) Riemannian, and 3) conformal geometry.

In my essay *The Nature of Time* [14], which won the first juried prize in the inaugural FQX essay competition, I showed how duration and the theory of clocks emerge from the *timeless* reparametrization-invariant Jacobi principle for the *orbit* of the Newtonian N-body problem:

$$\delta A_J = 0,\ A_J = 2\int \mathrm{d}\lambda \sqrt{(E-V)\sum_i \frac{m_i}{2}\,\mathbf{x}_i'\cdot\mathbf{x}_i'},\ \mathbf{x}_i' = \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}\lambda}, \tag{3}$$

where λ is an arbitrary monotonic parameter that labels the points of the orbit, E is the constant total energy, and V is the potential energy of the system. The two key features of (3) are: 1) the particle positions are the vectors $\mathbf{x}_i$ *in Euclidean space*, which will determine the nature of the emergent time; 2) the square root of the integrand, which makes the Lagrangian a metric on configuration space Q, the geodesics of which are the Newtonian orbits in Q with total energy E. The square-root structure ensures that there is no time in the kinematic foundations of the theory. Time only emerges as a parameter that simplifies the equation of the timeless geodesics that follows from (3) and is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sqrt{\frac{(E-V)}{T}}\,m_i\,\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}\lambda}\right) = -\sqrt{\frac{T}{(E-V)}}\frac{\partial V}{\partial \mathbf{x}_i}, \tag{4}$$

where T is the cofactor of $E-V$ in (3). It is obvious that (4) is greatly simplified by choosing the arbitrary label λ such that always $T = C(E-$

V), where the constant C sets the unit of the distinguished label λ, which then (since (4) then becomes Newton's second law) emerges as Newtonian time derived from change as Mach required. The explicit expression for the increment δt of this emergent time is

$$\delta t = \sqrt{\frac{\sum_i m_i \delta \mathbf{x}_i \cdot \delta \mathbf{x}_i}{2(E-V)}}. \tag{5}$$

It is shown in [14] that mechanical clocks will only march in step – and hence have any utility – if they are constructed in such a way that they measure this emergent time defined by (5). Jacobi's principle, properly interpreted for a closed system, provides *the complete theory of Newtonian time*. Moreover, the form of the expression (5) depends crucially on the properties of Euclidean space, which dictate the appearance of the scalar product in the numerator and the distance dependence that appears in V. In this sense, the nature of time is determined in dynamics by the structure of space.

4.2 The Derivation of Time with Riemannian Geometry

In general relativity, the situation is closely analogous but much more sophisticated. This is largely because space no longer has a Euclidean but rather the much richer Riemannian structure. The nature of the emergent time is correspondingly richer.

Vacuum general relativity (without cosmological constant and for a spatially closed universe) can be derived from the Baierlein–Sharp–Wheeler action [15]

$$A_{BSW} = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{gRG^{ijkl}(g'_{ij} - \xi_{(i;j)})(g'_{kl} - \xi_{(k;l)})},\ g'_{ij} := \frac{\mathrm{d}g_{ij}}{\mathrm{d}\lambda}, \tag{6}$$

where g is the determinant of the three-metric g_{ij}, R is the three-dimensional scalar curvature, G^{ijkl} is the DeWitt supermetric, and the three-vector field ξ_i in the Killing form $\xi_{(i;j)}$ is the generator of three-diffeomorphisms used in best matching to implement relativity of position in geometrodynamics. The BSW action (6) is defined on Riem, the space of Riemannian three-metrics on a closed three-manifold M. Variation with respect to ξ_i leads to the ADM momentum constraint. The parameter λ is arbitrary and in conjunction with the square root ensures that the kinematic foundation of the theory is timeless.

It is important to note that *the square root is taken locally*, i.e., a quadratic expression is formed at each space point and then its square root is integrated over space.[5] This has very significant consequences.

[5]For this reason, the BSW action, unlike the Jacobi action, cannot be regarded as a true geodesic principle though, as we shall see, it has the same effect. I should also express

First, the local square root means that the equations of motion that follow from (6), given in full in [2], simplify by analogy with what happens for Jacobi's principle if the value of λ is chosen *locally* such that

$$N = \sqrt{\frac{T}{4R}} = 1, \tag{7}$$

where T is the cofactor of $\sqrt{g}R$ in (6). This then leads to an explicit expression for an emergent *local* proper time. The richer structure of Riemannian geometry as compared with Euclidean geometry therefore has the consequence that its dynamics leads to a more subtle nature of time.

Second, because the square root is local, it leads to a primary quadratic constraint

$$p_{ij}p^{ij} - \frac{1}{2}p^2 - gR = 0,\ p = g_{ij}p^{ij}, \tag{8}$$

which is just the ADM Hamiltonian constraint derived directly from explicitly relational principles. As Ó Murchadha noted, it is extremely difficult to construct *consistent* theories of the form (6) with local square root. This has the positive consequences found in [2] and noted at the end of Sec. (3.2). In my view, they completely change the manner in which general relativity is to be viewed. There is at the least a strong case for regarding it as a relational dynamical theory, not a spacetime theory. The relativity of duration, implemented through the local square root, has numerous consequences and leads to the emergence of *local proper time* by very close analogy with the emergence of Newtonian time from the Jacobi action principle (3). The presence of the Killing term $\xi_{(i;j)}$ in (6) implements relativity of position and leads to the ADM momentum constraint. The fundamental dynamical structure of general relativity is derived very directly, as explained in detail in [2]. At this stage, we have relativity of both duration and simultaneity.

4.3 The Derivation of Time with Conformal Geometry

I now come to the most intriguing possibilities, which are opened up by implementing relativity of size in a spatially closed universe. As shown in [3, 4], this can be done by including in the BSW action (6) not only the Killing term $\xi_{(i;j)}$, which implements relativity of position, but a further term that implements relativity of size by best matching with respect to either full conformal transformations of the three-metric g_{ij} in accordance with (the fourth power of φ is chosen for mathematical convenience)

$$g_{ij} \to \varphi^4 g_{ij},\ \varphi = \varphi(x, \lambda) > 0, \tag{9}$$

my gratitude to Karel Kuchař for drawing my attention in 1980 to the local square root in the BSW action and its relation to Jacobi's principle. This feature had not been noted by any other relativist; its has been crucial for all recent progress in the relational program.

for such transformations that 1) are untrestricted or 2) preserve the total volume of the universe. The first possibility leads to a theory that is very similar to general relativity but in which no expansion of the universe is possible, the second to general relativity in the *constant-mean-(extrinsic)-curvature* (CMC) foliation that plays the key role in York's solution of the initial-value problem. The conformal best matching is implemented by varying φ by the free-end-point method (for details see [3, 4]) and, in contrast to the diffeomorphism best matching, leads to not only a constraint but also to a nontrivial consistency condition that ensures propagation of the constraint. The constraint enforces the CMC condition on the initial hypersurface, while the consistency condition, which is a lapse-fixing condition, ensures that this condition propagates.

These results are intriguing for two main reasons. First, a local scale appears nowhere in the kinematic foundations of the theory in conformal superspace but is forced to emerge as a distinguished scale in Riem by the best matching. This already happens on the initial hypersurface and is in strong contrast to the consequence of the diffeomorphism best matching, which does not lead to distinguished *coordinates* on the initial hypersurface but only in the propagation, for which Gaussian normal coordinates are distinguished. Second, the CMC condition, which introduces a *unique definition of simultaneity*, is enforced dynamically by the variational principle. Despite this, the effective theory that emerges at the *classical* level is identical to general relativity in its predictions. It may also be noted that there are an infinity of solutions of general relativity for which there exists at least a 'slab' of spacetime satisfying Einstein's equations in the CMC foliation. This follows from the very general applicability of York's method for solving the initial-value problem of general relativity in the CMC foliation and the existence and uniqueness of the solution of the CMC lapse-fixing equation, which ensures propagation of the condition in at least an open neighborhood of the initial hypersurface. It may be mentioned that the CMC simultaneity hypersurfaces in spacetime are analogous to soap bubbles in ordinary space and therefore have extremal properties.

The reason for the difference between the diffeomorphism and conformal constraints, which could have considerable significance for quantum gravity and leads to these two striking features, comes from a key property of what I call the 'bare' BSW action:

$$A_{bareBSW} = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{gRG^{ijkl}g'_{ij}g'_{kl}}, \tag{10}$$

which is simply the BSW action without the Killing term that implements diffeomorphism best matching. Now this action is invariant under *identical* diffeomorphisms made at each λ: it is *globally* gauge invariant, essentially because R is a scalar under diffeomorphisms. However, because gR is not a conformal scalar, the action (10) is not globally conformally invariant. In standard gauge theory with fixed-end-point variation, this would be regarded as a fatal defect but with the free-end-point variation that is appropriate for relational theories a perfectly consistent theory with very interesting properties arises. It is general relativity in the CMC foliation with the two angle-determing conformal degrees of freedom unambiguously identified as the true degrees of freedom of the theory.

One can summarize the situation as follows. One starts with *far less* kinematic structure than the standard approach; the assumptions about what exists are reduced to the absolute minimum. There is still a 'structured something' that evolves, but much of what is taken to exist in the standard view is not really present at all. In fact, all that exists in the classical theory is a sequence of conformal three-geometries. However, the manner in which the sequence is determined by best matching from specification of a point and tangent vector in CS makes it possible to embed the sequence of conformal three-geometries into an Einsteinian spacetime in the CMC foliation. Local inertial frames of reference, local proper distance, and local proper time are all emergent. In the spacetime representation, this 'additional structure' is assumed in to have ontological status but is in fact nomological in nature, i.e., it is an expression of the fundamental dynamical law of the universe and not part of its 'substance' at all.

This is very relevant to basic questions in quantum gravity. As a first example, serious attempts were made in the past to separate out from the full set of components of the four-metric $g_{\mu\nu}$ two true degrees of freedom from a 'residual structure' constituted by the remining components. This residual structure would then be a spatiotemporal framework in which the true degrees of freedom should evolve. But if, as the relational work strongly suggests, the 'residual framework' is not there at all, we must think about these things differently. The quantum universe will not evolve in any framework but simply be. As another example, the widespread assumption that space becomes discrete at the Planck length and may only exist in Planck-scale 'pieces' is perhaps natural if the true ontological basis of classical general relativity is Riemannian geometry. But if it is conformal geometry, that assumption becomes questionable. As one last example, the causal set program, while seeking, like the relational program, to identify the minimal structure needed to describe the quantum universe, looks to extract it from the four-dimensional causal structure of spacetime. But in the relational approach, this too is emergent.

Of course, I cannot claim that the relational approach is definitely correct. That would be rash. However, it does have very plausible first principles, is a radical alternative (with high-impact potential) to all the existing programs, and is based on some very solid mathematics (uniqueness and existence of the two key equations in the conformal approach: the Lichnerowicz–York equation and the lapse-fixing equation).

5 The Topics for Research

The relational program has consistently led to unexpected developments and I fully expect more topics for research to emerge as work progresses. At the moment, I plan four specific topics for research with the theoreticians mentioned already and below. Because their positions during the grant period are not yet known, I am not including them as formal co-investigators but do expect to be working with them. I may also mention that these topics are more precise than those listed for my 2008–2010 grant project (Machian Quantum Gravity) and emerged just as I had hoped from the mini-workshops and one-on-one interactions funded by it.

1. How Many Relational Theories Exist on Conformal Superspace? One of the issues that I am currently researching with Henrique Gomes (Nottingham University) and Sean Gryb (Perimeter Institute) and will continue in the period for the grant beginning January 2011 is that of how many theories exist on conformal superspace which meet the above 'point-and-tangent-vector' requirement. It has already been established that general relativity is such a theory. In principle, this was already shown in [4], but the recent research with Ó Murchadha has significantly clarified the derivation and made it more precise. However, general relativity is probably a special theory among a class of such theories, and it would be desirable to have an overview of the complete class. This will make it possible to understand better how general relativity can simultaneously have two such different interpretations (spacetime and relational) and is likely to be important in quantum gravity.

2. The Inclusion of Matter. The results so far described have all been for vacuum general relativity. An important topic for research will be to extend the conformal theory to include the matter fields, or at least those that are known to exist in nature. Because York's conformal method for solving the initial-value problem in general relativity was successfully extended to include matter already in the 1970s, it seems very likely that matter can be accommodated within the variational relational framework (which provides a first-principles *derivation* of York's method [4]). However, the details need to be worked out.

3. The Deep Dynamical Structure of General Relativity. Mention was made above of the very solid mathematics associated with York's conformal work. It is already evident that this is intimately related to the presence of the local square root in the BSW action, which in turn is what makes it possible to 'swap foliation invariance for three-dimensional conformal invariance'. Work on understanding this in depth and relating it to Dirac's theory of constrained Hamiltonian systems, which is important for quantum gravity, has already begun with Ó Murchadha, Gomes, Gryb, and

Tim Koslowski (Perimeter Institute). This work too I plan to continue in the period of the grant. It promises to give real insight into the structure of general relativity as a dynamical relational theory resting on a few key axioms.

4. The Relational Foundation of Gauge Theory. All this work belongs in the larger perspective of dynamical theories that are best represented in terms of fibre bundles in which the base space is a quotient configuration space obtained by quotienting with respect to a fundamental structure group. Without my realizing it, this structure was already implicit in my first paper [9]. I made the fibre-bundle picture the central theme in my Dennis Sciama Memorial Lecture ("Mach's principle as the universal basis of dynamics") given in Oxford and Trieste in November 2009. I pointed out that the relational perspective creates a deep unifying framework for (and explanation of) all gauge-type theories. Moreover, in conformal geometrodynamics the relational approach shows how gauge theory can be generalized in a highly nontrivial way using the variational procedure that Ó Murchadha introduced. I plan to make a significant part of my research to be the elaboration of the details in collaboration with Ó Murchadha, Gomes, Gryb, and Koslowski.

6 Outline of Proposed Book

I plan to devote half of the time of grant period to the writing of a book (provisional title: *The Principles of Relational Dynamics*). As I mentioned in the Introduction, it will be less historical and much more concerned with cutting edge research than its precursor [7], reprinted as [16]. I feel that my lifetime's work on the Machian approach has uniquely equipped me to contribute to clarifying many conceptual issues that are often very poorly understood. These include such basic issues as how to define inertial systems, clocks and duration. They are crucial to understanding the true relational nature of general relativity. In fact, the book will be a complete relational reworking of dynamics that frees it from all vestiges of the absolute structures that hinder the creation of quantum gravity. The earlier part of this proposal will have given a good idea of the subject matter of the book and its potential significance, so I here merely list some of the topics to be included:

Newton's reason's for introducing absolute space and time. The early criticism by Leibniz. Mach's critique. Poincaré's precise formulation of what a relational theory should achieve. The data and method needed to determine inertial frames of reference. The variational framework needed for relational dynamics: quotient configuration spaces and fibre bundles. Timless formulation of dynamics using Jacobi's principle and generalizations of it. The theory of time and clocks in particle dynamics. Scale invariance in particle dynamics. Early Machian theories. Best matching: the intuitive idea and the formal theory. Best matching in particle dynamics. General covariance and background independence. Noether's theorem and Dirac's generalized Hamiltonian theory. Maxwellian electrodynamics as treated by Dirac and as a best-matched relational theory. Relational geometrodynamics based on Riemannian geometry: the relational derivation of general relativity, then special relativity, and then gauge theory. Relational geometrodynamics based on conformal geometry. Implications of the relational structure of classical general relativity for quantum gravity.

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